PATH INTEGRAL ANALYIS FOR TIME-DEPENDENT HARMONIC OSCILLATORS

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Abstract

We demonstrate that the use of a space-time transformation in path integral can simplify the calculation of the propagator for a harmonic oscillator with time-dependent mass and frequency. We show that such a propagator can be easily obtained from the unit mass and frequency propagator in the new space-time coordinate systems. Two cases of harmonic oscillator with time-dependent mass, which exact propagator can be evaluated, are investigated.

Introduction

The Feynman path integral formulation of quantum mechanics provides an approach to solve quantum mechanical problems, alternative to the well-known formulations of Heisenberg and Schrodinger. The application of this method has been limited because explicit expressions for propagator are available only a few cases.

Recently there has been considerable interest in investigating the theory of time-dependent Hamiltonian systems using various methods. [1-5] Various application in many areas of physics, such as quantum optics, cosmology, and nanotechnology are the main reasons for intensive study. S. Pepore and et.al. [6-8] applied both Feynman path integral and Schwinger method to study the propagator and wave function for a harmonic oscillator with time-dependent mass and frequency.

The aim of this paper is to derive the propagator for a harmonic oscillator with time-dependent mass and frequency as described by the Hamiltonian

$$H(t)=\frac{p^2}{2m(t)}+\frac{1}{2}m(t)\omega^2(t)x^2,$$

where $m(t)$ and $\omega(t)$ are the time-dependent mass and frequency, respectively.

Our method is not based on the direct calculating of path integration, but based on the using of a space-time transformation to simplify the path integration. We also present two more exactly solvable cases of harmonic oscillators with time-dependent mass and constant frequency:

1. A damping mass obeying $m(t)=me^{rt}$
2. A strongly pulsating mass according to $m(t)=m\cos^2 \nu t$.

We have calculated exact closed from expressions for the propagators for these two cases.

The Space-Time Transformation in Path Integration for a Harmonic Oscillator with Time-Dependent Mass and Frequency

The dynamics of a harmonic oscillator with time-dependent mass and frequency can be described by the Lagrangian [6] as

$$L(t) = \frac{1}{2} m(t) \dot{x}^2 - \frac{1}{2} m(t) \omega^2(t) x^2 , \quad (1)$$

where $m(t)$ is the time-dependent mass and $\omega(t)$ is the time-dependent frequency. By using the Euler-Lagrange equation for the Lagrangian in Eq.(1), the equation of motion can be written as

$$\ddot{x} + 2\frac{\dot{\eta}}{\eta}\dot{x} + \omega^2(t) x = 0 , \quad (2)$$

where we define $\eta(t) = \sqrt{m(t)}$.

By using the Pinney equation [9]

$$\ddot{\rho} + \frac{\dot{m}(t)}{m(t)}\dot{\rho}(t) + \omega^2(t)\rho(t) = \frac{1}{m^2(t)\rho^3(t)} , \quad (3)$$

the Lagrangian in Eq.(1) can be modified to

$$L(t) = \frac{d}{dt}\left(\frac{m\dot{\rho}x^2}{2\rho}\right) + L_0 , \quad (4)$$

where L_0 is

$$L_0 = \frac{1}{2} m^2 \left(\dot{x}\rho - \dot{\rho}x\right)^2 - \frac{1}{2}\left(\frac{x}{\rho}\right)^2 . \quad (5)$$

The next step is trying to find a transformation that can transform the system with Lagrangian L_0 in Eq. (5) into the harmonic oscillator with unit mass and frequency. Let us consider the following transformation, [10] which is the space and time transformation,

$$y(t) = \frac{x(t)}{\rho(t)} , \quad (6)$$

and

$$d\tau = \frac{dt}{m(t)\rho^2(t)} . \quad (7)$$

By using space and time transformation, the Lagrangian L_0 in Eq. (5) can be written as

$$\bar{L}_0 = \frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 - \frac{1}{2} y^2 . \quad (8)$$

The Feynman propagator $K(x'',t'';x',t')$ is defined as the path integral [11]

$$K(x'',t'';x',t') = \int \exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L dt\right) Dx(t) , \quad (9)$$

where Dx(t) is the path differential measure indicating that integrations are over all possible paths beginning at $x(t') = x'$ and terminating at

$$x(t'') = x'' .$$

By substituting the Lagrangian in Eq. (4) into Eq. (9), the propagator can be expressed as

$$K(x'',t'';x',t') = K_0 \exp\left\{\frac{i}{2\hbar}\left[\frac{m''\dot{\rho}''}{\rho''}x''^2 - \frac{m'\dot{\rho}'}{\rho'}x'^2\right]\right\},$$

(10)

where K_0 is the new propagator corresponding to the new Lagrangian

$$K_0 = \int \exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L_0 dt\right) Dx(t). \quad (11)$$

If we now introduce a new time τ in Eq.(7)

$$\tau(t) = \int^{t} \frac{1}{m(s)\rho^2(s)} ds, \quad (12)$$

the action integral in Eq. (11) takes the form

$$\int_{t'}^{t''} L_0 dt = \int_{\tau'}^{\tau''} \overline{L}_0 d\tau, \quad (13)$$

where $\overline{L}_0$ is the unit mass and frequency oscillator Lagrangian in Eq. (8).

Using a process similar to Lawande and Dhara, [10] the transformation of the measures can be expressed as

$$Dx(t) = \frac{1}{\sqrt{\rho'\rho''}} Dy(\tau). \quad (14)$$

So, the propagator in Eq. (10) can be written as

$$K(x'',t'';x',t') = \frac{1}{\sqrt{\rho'\rho''}}$$

$$\exp\left\{\frac{i}{2\hbar}\left[\frac{m''\dot{\rho}''x''^2}{\rho''} - \frac{m'\dot{\rho}'x'^2}{\rho'}\right]\right\} K_0(y'',\tau'';y',\tau'), \quad (15)$$

where $K_0(y'',\tau'';y',\tau')$ is the propagator for a harmonic oscillator with unit mass and frequency described by [11]

$$K_0(y'',\tau'';y',\tau') = \int \exp\left(\frac{i}{\hbar}\int_{\tau'}^{\tau''} \overline{L}_0 d\tau\right) Dy(\tau)$$

$$= \left(\frac{1}{2\pi i\hbar \sin(\tau''-\tau')}\right)^{\frac{1}{2}}$$

$$\exp\left\{\frac{i}{2\hbar \sin(\tau''-\tau')}\left[(y''^2 + y'^2)\cos(\tau''-\tau') - 2y'y''\right]\right\}. \quad (16)$$

Substituting Eq. (16) into Eq.(15), the result is

$$K(x'',t'';x',t') = \left(\frac{1}{2\pi i\hbar \rho'\rho'' \sin(\tau''-\tau')}\right)^{\frac{1}{2}}$$

$$\exp\left\{\frac{i}{2\hbar}\left[\frac{m''\dot{\rho}''x''^2}{\rho''} - \frac{m'\dot{\rho}'x'^2}{\rho'}\right]\right\}$$

$$\times \exp\left\{\frac{i}{2\hbar \sin(\tau''-\tau')}\left[(y''^2 + y'^2)\cos(\tau''-\tau') - 2y'y''\right]\right\}. \quad (17)$$

The final step is rewriting Eq. (17) into the original variables as

$$K(x'',t'';x',t') = \left(\frac{1}{2\pi i\hbar y'y'' \sin(\tau''-\tau')}\right)^{\frac{1}{2}}$$

$$\exp\left\{\frac{i}{2\hbar}\left[\frac{m''\dot{\rho}''x''^2}{\rho''} - \frac{m'\dot{\rho}'x'^2}{\rho'}\right]\right\}$$

$$\times \exp\left[\frac{i}{2\hbar \sin(\tau''-\tau')}\left\{\left(\frac{x''^2}{\rho''^2} + \frac{x'^2}{\rho'^2}\right)\cos(\tau''-\tau') - \frac{2x'x''}{\rho'\rho''}\right\}\right]. \quad (18)$$

This result is agree with the result of S.Pepore and B.Sukbot by using of the Schwinger method. [7]

The Caldirola-Kanai Oscillator

In this section, the application of the solution of our auxiliary Eq. (3) and (7) is demonstrated to derive the exact propagator. The system selectedas an example is the damped harmonic oscillator or the Caldirola-Kanai oscillator. [12-13]

The time-dependent mass for a damped harmonic oscillator can be written as

$$m(t) = me^{rt}, \quad (19)$$

where m is the constant mass and r is the constant damping coefficient. The Caldirola-Kanai Lagrangian can be obtained by the Lagrangian in Eq. (1) with constant frequency ω

$$L(t) = \frac{1}{2}me^{rt}\dot{x}^2 - \frac{1}{2}me^{rt}\omega^2 x^2. \quad (20)$$

In order to obtain the propagator of the system, the explicit forms of the functions $\rho(t)$

in Eq.(3) and $\tau(t)$ in Eq.(12) have to be solved. By substituting the time-dependent mass in Eq.(19) into the Pinney equation Eq. (3) and Eq. (12), it can be derived that

$$\rho(t)=\frac{e^{-rt/2}}{\sqrt{m\Omega}} \tag{21}$$

and

$$\tau(t)=\Omega t\,, \tag{22}$$

where Ω is the reduced frequency defined by

$$\Omega^2=\omega^2-\frac{r^2}{4}\,. \tag{23}$$

By substituting Eqs. (19), (21) and (22) into Eq. (18), the propagator for the Caldirola-Kanai oscillator can be obtained as

$$K(x'',t'';x',t')=\left(\frac{m\Omega e^{r(t''+t')/2}}{2\pi i\hbar\sin\Omega(t''-t')}\right)^{\frac{1}{2}}$$

$$\exp\left\{\frac{im\Omega}{2\hbar}\left[\cot\Omega(t''-t')\left(e^{rt''}x''^2+e^{rt'}x'^2\right)-\frac{2x'x''e^{r(t''+t')/2}}{\sin\Omega(t''-t')}\right]\right\}$$

$$\times\exp\left\{\frac{imr}{4\hbar}(e^{rt''}x''^2-e^{rt'}x'^2)\right\}. \tag{24}$$

The obtained propagator in Eq. (24) are in the same from as that reported by Jannusis and et.al. [12]

The Harmonic Oscillator with Strongly Pulsating Mass

The other well known of a time-dependent mass oscillator is a harmonic oscillator with strongly pulsating mass. [14] This oscillator can be applied in connection with the electromagnetic field in a Fabry-Perot cavity in contact with a reservoir of resonant two-level atoms. The periodic release and reabsorption of photon can be represented by an oscillator of periodically fluctuating energy. In other words, it can be represented by a periodically varying mass as

$$m(t)=m\cos^2\nu t, \tag{25}$$

where ν is the frequency of mass. In this case the Lagrangian becomes

$$L(t)=\frac{1}{2}m\cos^2\nu t\dot{x}^2-\frac{1}{2}m\cos^2\nu t\omega^2x^2. \tag{26}$$

By substituting the mass law into the auxiliary Eq. (3) Eq. (12), we can get

$$\rho(t)=\frac{\sec\nu t}{\sqrt{m\Omega}} \tag{27}$$

and

$$\tau(t)=\Omega t\,, \tag{28}$$

where the augmented frequency Ω is defined by

$$\Omega^2=\omega^2+\nu^2. \tag{29}$$

By substituting Eqs.(25), (27), and (28) into the propagator in Eq. (18), the propagator for a harmonic oscillator with strongly pulsating mass can be derived as

$$K(x'',t'';x',t')=\left(\frac{m\Omega\cos\nu t'\cos\nu t''}{2\pi i\hbar\sin\Omega(t''-t')}\right)^{\frac{1}{2}}$$

$$\exp\left[\frac{im\nu}{2\hbar}\left(\cos^2\nu t''\tan\nu t''x''^2-\cos^2\nu t'\tan\nu t'x'^2\right)\right]$$

$$\times\exp\left(\frac{im\Omega}{2\hbar\sin\Omega(t''-t')}\left[\left(\cos^2\nu t''x''^2+\cos^2\nu t'x'^2\right)\cos\Omega(t''-t')-2\cos\nu t'\cos\nu t''x'x''\right]\right). \tag{30}$$

This propagator can be simplified by setting $\nu=0$, and $\Omega=\omega$. The result is reduced to the simple harmonic oscillator propagator.

Conclusion

In this article we have successfully calculated the exact propagator for a harmonic oscillator with time-dependent mass and frequency by the Feynman path integral method in combination with a space-time transformation. The resulting propagator in Eq. (18) is similar to as in the report of Pepore and et.al. [6-8] An important step in this paper is to find the space and time transformation in Eq. (6) and Eq. (7) and to write the Lagrangian in terms of a unit mass and frequency oscillator in Eq. (8). The advantage of our method is that it can transform complicated system into a simplified problem. We have concluded here that, our approach is an effective method in solving the time-dependent problems because it only requires some basic integration. In section 3 and 4, we have shown the usefulness of Pinney equation for deriving the explicit form of the propagator in the case of the Caldirola-Kanai and strongly pulsating mass oscillator. Finally, it may be suggested that the methods in this paper can be applied to complicated problems, such as a time-dependent linear potential and a charged harmonic oscillator in a time-dependent electromagnetic field.

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