Schwinger Method for Damped Mechanical Systems

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ABSTRACT

The propagators for two damped mechanical systems are exactly solved by the Schwinger method. The first system is a free particle in linear damping, while the second system is a particle in a uniform gravitational field with linear damped motion. The propagators can be obtained by basic operator algebra and elementary integrations. The power of the Schwinger method in solving non-relativistic quantum systems is also discussed.

Keywords: Schwinger method, Propagator, Damped mechanical system

Introduction

The presence of irreversible, dissipative behavior of physical phenomena is not as easy to understand. The well-known model to study damped mechanical systems is a damped harmonic oscillator [1-2]. Bateman [3] proposed the time-dependent Hamiltonian in the classical context for the description of dissipative systems. To investigate the quantum mechanical description of damped mechanical systems, there may be studied by Caldirola-Kanai Hamiltonian [4-5].

Recently, D. Kochan [6] apply functional integral or stringy quantization to compare the approaches of Caldirola-Kanai, and Bateman. D. Jain and et.al. employs path integrals to derive the propagators for the linearly damped harmonic oscillator and a particle in a uniform gravitational field with linearly or quadratically damped motion [7]. Altarawneh and Hasan quantize a quadratic damped mechanical system using the WKB formalism [8]. The aim of this paper is to calculate the propagators for a free particle in linear damping and a particle in a uniform gravitational field with linearly damped by Schwinger method [9].

Recently, there has been a report on the calculating of non-relativistic propagators by the Schwinger method [10-19]. This method was first formulated by Schwinger in 1951 for solving the gauge invariance and vacuum polarization in QED [9]. The Schwinger method was used to derive the propagator for a damped harmonic oscillator with a time-dependent frequency under a time-dependent external force by Urrutia and Hernandez [10-11]. In 1986, Horing, Cui, and Fiorenza applied this method to calculate the propagator for a charged particle in crossed time-dependent electric and magnetic fields

[12]. Aragao et. al. [13] used the Schwinger method to obtain the propagator for a charged particle in a uniform magnetic field in 2007. Finally, Pepore and Sukbot applied this method to evaluate the propagator for quadratic Hamiltonian systems both time-dependent and time-independent [16-19].

In section 2, the procedures of the Schwinger method are described. In section 3, the propagator for a free-particle with linear damping are derived by the Schwinger method. In section 4, the propagator for a particle moving in a uniform gravitational field with linear damping is calculated. Finally, the conclusion and discussion are given in section 5.

Procedures of the Schwinger Method

Begin by considering a time-dependent Hamiltonian H(t); the propagator is defined by [22]

$$K(x,\tau;x',0) = < x|\hat{T}\exp(-\tfrac{i}{\hbar}\int_0^\tau \hat{H}(t)dt)|x' >, \tag{1}$$

where $\hat{T}$ is time-ordering operator and $|x>$, $|x'>$ are the eigenvectors of the position operator $\hat{x}$ (in the Schrodinger picture) with eigenvalues x and x', respectively.

The differential equation for the propagator in Eq. (1) can be written as

$$i\hbar\frac{\partial}{\partial\tau}K(x,\tau;x',0) = < x|\hat{T}\hat{H}\exp(-\tfrac{i}{\hbar}\int_0^\tau \hat{H}(t)dt)|x' >. \tag{2}$$

Using the relation between operators in the Heisenberg and Schrodinger pictures, we get the equation for the propagator in the Heisenberg picture:

$$i\hbar\frac{\partial}{\partial\tau}K(x,\tau;x',0) = < x(\tau)\big|\hat{H}\big(\hat{x}(\tau),\hat{p}(\tau)\big)\big|x'(0) >, \tag{3}$$

where $|x(\tau)>$ and $|x'(0)>$ are the eigenvectors of the operators $\hat{x}(\tau)$ and $\hat{x}(0)$, respectively, with the corresponding eigenvalues x and x'. Besides, $\hat{x}(\tau)$ and $\hat{p}(\tau)$ satisfy the Heisenberg equations:

$$i\hbar\frac{d\hat{x}(\tau)}{d\tau} = \big[\hat{x}(\tau),\hat{H}\big], \qquad i\hbar\frac{d\hat{p}(\tau)}{d\tau} = \big[\hat{p}(\tau),\hat{H}\big]. \tag{4}$$

The main idea of the Schwinger method consists in the following steps.

(1). The first step is solving the Heisenberg equations for $\hat{x}(\tau)$ and $\hat{p}(\tau)$, and writing the solution for $\hat{p}(\tau)$ only in terms of the operators $\hat{x}(\tau)$ and $\hat{x}(0)$.

(2). Next, we substitute the solutions obtained in step (1) into the expression for $\hat{H}(\hat{x}(\tau),\hat{p}(\tau))$ in Eq. (3) and use the commutator $[\hat{x}(0),\hat{x}(\tau)]$ to rewrite each term of $\hat{H}$ in a time ordered form with all operators $\hat{x}(\tau)$ to the left and all operators $\hat{x}(0)$ to the right.

(3). After this ordering, Eq. (3) can be written in the form

$$i\hbar\frac{\partial}{\partial\tau}K(x,\tau;x',0) = H(x,\tau;x',0)K(x,\tau;x',0), \tag{5}$$

with $H(x,\tau;x';0)$ being an ordering function defined as

$$H(x,\tau;x',0) = \frac{<x(\tau)|\hat{H}_{ord}(\hat{x}(\tau),\hat{x}(0)|x'(0)>}{<x(\tau)|x'(0)>}. \tag{6}$$

Integrating Eq. (5) over τ, the propagator takes the form

$$K(x,\tau;x',0) = C(x,x')\exp\{-\tfrac{i}{\hbar}\int_0^\tau H(x,t;x',0)dt\}, \tag{7}$$

where $C(x, x')$ is an integration constant.

(4). The last step is the calculation of $C(x, x')$. This is obtained by using the following conditions:

$$-i\hbar\frac{\partial}{\partial x}K(x,\tau;x',0) = < x(\tau)|\hat{p}(\tau)|x'(0) >, \tag{8}$$

$$i\hbar\frac{\partial}{\partial x'}K(x,\tau;x',0) = < x(\tau)|\hat{p}(0)|x'(0) >, \tag{9}$$

and the initial condition

$$\lim_{\tau\to 0^+} K(x,\tau;x',0) = \delta(x-x'). \tag{10}$$

The Propagator for a Free Particle with Linear Damping

This section is the calculation of the propagator for a free particle with linear damping by the Schwinger method. Considering the motion of a free particle with constant mass m in a linear damping which has the damping coefficient β, the Hamiltonian of this system can be written as [7]

$$H(\tau) = \frac{p^2(\tau)}{2m}e^{-\gamma\tau}, \tag{11}$$

where $\gamma = \frac{\beta}{m}$.

The equation of motion corresponding to the Hamiltonian in Eq. (11) is

$$m\ddot{x} + \beta\dot{x} = 0. \tag{12}$$

The classical solution of Eq. (12) is given by

$$x(\tau) = A + Be^{-\gamma\tau}, \tag{13}$$

where A and B are constants.

The constants A and B in Eq. (13) can be determined by imposing the initial conditions $x(0) = x'$ and $p(0) = p'$. After some algebraic manipulations it can be shown that

$$x(\tau) = x' + \frac{(1-e^{-\gamma\tau})}{m\gamma}p'. \tag{14}$$

By solving the Heisenberg operator equation in Eq. (4), the position operator $\hat{x}(\tau)$ can be written similarly to Eq. (14) as

$$\hat{x}(\tau) = \hat{x}(0) + \frac{(1-e^{-\gamma\tau})}{m\gamma}\hat{p}(0). \tag{15}$$

The momentum operator $\hat{p}(\tau) = m\dot{\hat{x}}(\tau)e^{\gamma\tau} = \hat{p}(0)$ can be written by using Eq. (15) as

$$\hat{p}(\tau) = \frac{m\gamma}{(1-e^{-\gamma\tau})}(\hat{x}(\tau) - \hat{x}(0)). \tag{16}$$

Substituting $\hat{p}(\tau)$ into the Hamiltonian operator $\widehat{H}(\tau) = \frac{\hat{p}^2(\tau)}{2m}e^{-\gamma\tau}$, the result is

$$\widehat{H}(\tau) = \frac{m\gamma^2 e^{-\gamma\tau}}{2(1-e^{-\gamma\tau})^2}[\hat{x}^2(\tau) - \hat{x}(\tau)\hat{x}(0) - \hat{x}(0)\hat{x}(\tau) + \hat{x}^2(0)], \tag{17}$$

where this Hamiltonian operator can be written in an appropriate operator form with the helping of the commutator

$$[\hat{x}(0), \hat{x}(\tau)] = \frac{i\hbar}{m\gamma}(1 - e^{-\gamma\tau}). \tag{18}$$

The ordered Hamiltonian operator can be expressed as

$$\hat{H}_{ord}(\tau) = \frac{m\gamma^2 e^{-\gamma\tau}}{2(1-e^{-\gamma\tau})^2}\left[\hat{x}^2(\tau) - 2\hat{x}(\tau)\hat{x}(0) + \hat{x}^2(0)\right] - \frac{i\hbar\gamma e^{-\gamma\tau}}{2(1-e^{-\gamma\tau})}. \quad (19)$$

Using Eqs. (5) - (7), the propagator takes the form

$$K(x,\tau;x',0) = C(x,x')exp\left(\frac{-\frac{i}{\hbar}\int_0^\tau <x(t)|\hat{H}_{ord}(t)|x'(0)>dt}{<x(t)|x(0)>}\right)$$

$$= C(x,x')exp\left\{-\frac{i}{\hbar}\int_0^\tau\left(\frac{m\gamma^2 e^{-\gamma t}}{2(1-e^{-\gamma t})^2}(x-x')^2 - \frac{i\hbar\gamma e^{-\gamma t}}{2(1-e^{-\gamma t})}\right)dt\right\}. \quad (20)$$

Now, we will integrate each term of Eq. (20) with respect to time. The first term of Eq. (20) can be integrated as

$$-\frac{im\gamma^2(x-x')^2}{2\hbar}\int_0^\tau \frac{e^{-\gamma t}}{(1-e^{-\gamma t})}dt = \frac{im\gamma(x-x')^2}{2\hbar(1-e^{-\gamma t})}. \quad (21)$$

The second term of Eq. (20) can be calculated as

$$-\frac{\gamma}{2}\int_0^\tau \frac{e^{-\gamma t}}{(1-e^{-\gamma t})}dt = -\frac{1}{2}\ln(1-e^{-\gamma\tau}). \quad (22)$$

Combining the results in Eqs. (20)-(22), the propagator can be written as

$$(x,\tau;x',0) = \frac{C(x,x')}{\sqrt{1-e^{-\gamma\tau}}}exp\left(\frac{im\gamma(x-x')^2}{2\hbar(1-e^{-\gamma\tau})}\right). \quad (23)$$

The final step is finding the function $C(x,x')$ by substituting Eq. (23) into Eqs. (8) and (9). The results are

$$\frac{\partial}{\partial x}C(x,x') = 0, \quad (24)$$

and

$$\frac{\partial}{\partial x'}C(x,x') = 0, \quad (25)$$

which implies that $C(x,x')$ is a constant independent of x and x'.

After applying Eq. (10), it can be shown that

$$C(x,x') = \sqrt{\frac{m\gamma}{2\pi i\hbar}}. \quad (26)$$

So, the propagator for a free particle with linear damping can be written as

$$K(x,\tau;x',0) = \sqrt{\frac{m\gamma}{2\pi i\hbar(1-e^{-\gamma\tau})}}exp\left(\frac{im\gamma(x-x')^2}{2\hbar(1-e^{-\gamma\tau})}\right). \quad (27)$$

The Propagator for a Particle in a Uniform Gravitational Field with Linear Damping

This section is the calculation of the propagator for a particle in a uniform gravitational field with linear damping by the Schwinger action principle [20]. The Hamiltonian of this system can be described by [7]

$$H(t) = \frac{p^2}{2m}e^{-\gamma t} - mgxe^{\gamma t}, \quad (28)$$

where g is a constant gravitational field and $\gamma = \frac{\beta}{m}$.

Schwinger [20] introduced that any infinitesimal variation of the propagator $< 1|0 > \equiv < x, t|x_0, t_0 >$ can be expressed as

$$\delta < 1|0 >= \frac{i}{\hbar} < 1|\delta \widehat{W}_{10}|0 >, \tag{29}$$

where 1 and 0 stand for the set of quantum numbers that label the initial and final states and $\delta \widehat{W}_{10}$ is the infinitesimal action operator .

Let $\delta \widehat{W}'_{10}$ is the well-ordered form of $\delta \widehat{W}_{10}$. So

$$\delta < 1|0 > = \frac{i}{\hbar} < 1|\, \delta \widehat{W}'_{10}|0 > = \frac{i}{\hbar} \delta W'_{10} < 1|0 >, \tag{30}$$

Or
$$< 1|0 >= \exp(i\hbar^{-1} W'_{10}). \tag{31}$$

The classical Hamilton-Jacobi equation corresponding to the Hamiltonian in Eq. (28) can be written as [21]

$$\frac{e^{-\gamma t}}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - mgxe^{\gamma t} + \frac{\partial S}{\partial t} = 0. \tag{32}$$

The solution of Eq. (32) is

$$S_c = \frac{m\gamma e^{\gamma t}}{2(e^{\gamma t}-1)}\left[(x - x_0) - \frac{gt}{\gamma}\right]^2 + \frac{mg}{\gamma}(xe^{\gamma t} - x_0) + \frac{mg^2}{2\gamma^3}(1 - e^{-\gamma t}). \tag{33}$$

The corresponding operator equation of Eq. (32) is

$$\frac{e^{-\gamma t}}{2m}\left(\frac{\partial \widehat{W}}{\partial x}\right)^2 - mg\hat{x}e^{\gamma t} + \frac{\partial \widehat{W}}{\partial t} = 0. \tag{34}$$

For solving Eq. (34), let us write the Ansatz,

$$\widehat{W} = \hat{S} + \varphi(t), \tag{35}$$

where $\hat{S}$ is obtained from S_c by writing x to the left of x_0 in S_c,

$$\hat{S} = \frac{m\gamma e^{\gamma t}}{2(e^{\gamma t}-1)}(\hat{x}^2 + \hat{x}_0^2 - 2\hat{x}\hat{x}_0) - \frac{mgte^{\gamma t}}{(e^{\gamma t}-1)}(\hat{x} - \hat{x}_0) + \frac{mg^2t^2e^{\gamma t}}{2\gamma(e^{\gamma t}-1)}$$

$$+\frac{mg}{\gamma}(\hat{x}e^{\gamma t}-\hat{x}_0)+\frac{mg^2}{2\gamma^3}(1-e^{-\gamma t}), \tag{36}$$

and $\varphi(t)$ commutes with all operators, and vanishes in the classical limit. The momentum operators can be calculated from

$$\hat{p}=\frac{\partial\widehat{W}}{\partial\hat{x}}=\frac{m\gamma e^{\gamma t}}{(e^{\gamma t}-1)}(\hat{x}-\hat{x}_0)-\frac{mgte^{\gamma t}}{(e^{\gamma t}-1)}+\frac{mg}{\gamma}e^{\gamma t}\ . \tag{37}$$

Using the basic commutator $[\hat{x},\hat{p}]=i\hbar$ and Eq. (37), it can be shown that

$$[\hat{x},\hat{x}_0]=-\frac{i\hbar}{m\gamma}(1-e^{-\gamma t})\ . \tag{38}$$

To obtain an equation that is satisfied by $\varphi(t)$, let us calculate

$$\left(\frac{\partial\widehat{W}}{\partial\hat{x}}\right)^2=\frac{m^2\gamma^2e^{2\gamma t}}{(e^{\gamma t}-1)^2}(\hat{x}^2-\hat{x}\hat{x}_0-\hat{x}_0\hat{x}+\hat{x}_0^2)-\frac{2m^2g\gamma te^{2\gamma t}}{(e^{\gamma t}-1)^2}(\hat{x}-\hat{x}_0)+\frac{2m^2ge^{2\gamma t}}{(e^{\gamma t}-1)}(\hat{x}-\hat{x}_0)$$

$$+\frac{m^2g^2t^2e^{2\gamma t}}{(e^{\gamma t}-1)^2}-\frac{2m^2g^2te^{2\gamma t}}{\gamma(e^{\gamma t}-1)}+\frac{m^2g^2}{\gamma^2}e^{2\gamma t}\ . \tag{39}$$

By using the commutator in Eq. (38), we can rewrite Eq. (39) in the well-ordered form as

$$\left(\frac{\partial\widehat{W}}{\partial\hat{x}}\right)^2=\frac{m^2\gamma^2e^{2\gamma t}}{(e^{\gamma t}-1)^2}(\hat{x}^2-2\hat{x}\hat{x}_0+\hat{x}_0^2)-\frac{i\hbar m\gamma e^{\gamma t}}{(e^{\gamma t}-1)}-\frac{2m^2g\gamma te^{2\gamma t}}{(e^{\gamma t}-1)^2}(\hat{x}-\hat{x}_0)+\frac{2m^2ge^{2\gamma t}}{(e^{\gamma t}-1)}(\hat{x}-\hat{x}_0)$$

$$+\frac{m^2g^2t^2e^{2\gamma t}}{(e^{\gamma t}-1)^2}-\frac{2m^2g^2te^{2\gamma t}}{\gamma(e^{\gamma t}-1)}+\frac{m^2g^2}{\gamma^2}e^{2\gamma t}\ . \tag{40}$$

The next step is differentiating Eq. (36) with respect to time as

$$\frac{\partial\hat{S}}{\partial t}=-\frac{m\gamma^2e^{\gamma t}}{2(e^{\gamma t}-1)^2}(\hat{x}^2-2\hat{x}\hat{x}_0+\hat{x}_0^2)+\frac{mgt\gamma e^{\gamma t}}{(e^{\gamma t}-1)^2}(\hat{x}-\hat{x}_0)-\frac{mge^{\gamma t}}{(e^{\gamma t}-1)}(\hat{x}-\hat{x}_0)$$

$$-\frac{mg^2t^2e^{\gamma t}}{2(e^{\gamma t}-1)^2}+\frac{mg^2te^{\gamma t}}{\gamma(e^{\gamma t}-1)}+mge^{\gamma t}\hat{x}+\frac{mg^2}{2\gamma^2}e^{-\gamma t}\ . \tag{41}$$

An expression for the derivative of $\widehat{W}$ with respect to time in Eq. (35) can be obtained by using

$$\frac{\partial\widehat{W}}{\partial t}=\frac{\partial\hat{S}}{\partial t}+\frac{\partial\varphi}{\partial t}\ . \tag{42}$$

Substituting Eq. (40)-(42) into Eq. (34), it can be obtained that

$$\frac{\partial\varphi}{\partial t} = \frac{i\hbar\gamma}{2(e^{\gamma t}-1)} - \frac{mg^2}{2\gamma^2}(e^{\gamma t} + e^{-\gamma t}). \tag{43}$$

Integrating Eq. (43), the result is

$$\varphi(t) = \frac{i\hbar}{2}(ln|e^{\gamma t} - 1| - \gamma t + C) - \frac{m^2 g}{\gamma^3} sinh(\gamma t), \tag{44}$$

where C is a constant of integration.

By using Eq. (31), (33) and (44), the propagator takes the form

$$K(x,t;x_0,0) = exp\left(\frac{i}{\hbar}\varphi(t)\right) exp\left(\frac{i}{\hbar}S_c\right)$$
$$= C\sqrt{\frac{e^{\gamma t}}{e^{\gamma t}-1}} exp\left(\frac{i}{\hbar}\left\{\frac{m\gamma e^{\gamma t}}{2(e^{\gamma t}-1)}(x-x_0)^2 - \frac{mgte^{\gamma t}}{(e^{\gamma t}-1)}(x-x_0) + \frac{mg^2t^2e^{\gamma t}}{2\gamma(e^{\gamma t}-1)} + \frac{mg}{\gamma}(xe^{\gamma t} - x_0) + \frac{mg^2}{2\gamma^3}(1-e^{\gamma t})\right\}\right). \tag{45}$$

The constant C in Eq. (45) can be calculated by applying Eq. (10) as

$$C = \sqrt{\frac{m\gamma}{2\pi i\hbar}}. \tag{46}$$

So, the propagator for a particle in a uniform gravitational field with linear damping is

$$K(x,t;x_0,0) = \sqrt{\frac{m\gamma e^{\gamma t}}{2\pi i\hbar(e^{\gamma t}-1)}} exp\left(\frac{i}{\hbar}\left\{\frac{m\gamma e^{\gamma t}}{2(e^{\gamma t}-1)}(x-x_0)^2 - \frac{mgte^{\gamma t}}{(e^{\gamma t}-1)}(x-x_0) + \frac{mg^2t^2e^{\gamma t}}{2\gamma(e^{\gamma t}-1)} + \frac{mg}{\gamma}(xe^{\gamma t} - x_0) + \frac{mg^2}{2\gamma^3}(1-e^{\gamma t})\right\}\right). \tag{47}$$

This propagator is the same form as the calculation of D. Jain and et. al. [7] by Feynman path integral method. The obtained propagator in Eq. (47) can be reduced to the propagator in Eq. (27) by setting $g = 0.$

Conclusions

In this paper, we have successfully calculated the exact propagators for the free particle in linear damping and the particle in uniform gravitational field with linear damping. The resulting propagator in Eq. (47) is the same as in the report of Jain and et. al. [7] and can be reduced to Eq. (27) in the case of non-gravitational field.

The important step in the calculation of a propagator for a free particle in linear damping is to find the solutions of the Heisenberg equations in Eq. (15) and (16) and to write the Hamiltonian operator in an appropriate order with the help of the commutator in Eq. (18). The crucial step for the derivation of the propagator for a particle in a uniform gravitational field with linear damping is to find the solution of the Hamilton-Jacobi equation in Eq. (33) and to evaluate the phase factor in Eq. (44). The advantage of the Schwinger method in this paper is that it requires only fundamental operator algebra and some basic integration.

Finally, we have presented a simple technique for calculating quantum solutions. It is preferable to have many tools for attacking the problems in addition to the standard method of the Feynman path integral [22]. It may be suggested that the methods in this article can be generalized to others quadratic Hamiltonian systems, such as a particle in uniform gravitational field with quadratic damping.

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