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สมการไดโอแฟนไทน์ $3^x - p^y = z^2$ เมื่อ p เป็นจำนวนเฉพาะ
ON THE DIOPHANTINEEQUATION $3^x - p^y = z^2$ WHERE p IS PRIME

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บทคัดย่อ

ในงานวิจัยนี้ได้แสดงว่า $(x, y, z) = (0, 0, 0)$ เป็นผลเฉลยที่เป็นจำนวนเต็มที่ไม่เป็นลบเพียงผลเฉลยเดียวของสมการไดโอแฟนไทน์ $3^x - p^y = z^2$ เมื่อ p เป็นจำนวนเฉพาะ และ x, y, z เป็นจำนวนเต็มที่ไม่เป็นลบ โดยมีเงื่อนไขบางประการ และสำหรับกรณี $p = 2$ จะให้ผลเฉลยที่เป็นจำนวนเต็มที่ไม่เป็นลบทั้งหมดของสมการดังกล่าว

คำสำคัญ: สมการไดโอแฟนไทน์ ทฤษฎีของมิเฮเลสคู ผลเฉลยที่เป็นจำนวนเต็มที่ไม่เป็นลบ

Abstract

In this paper, we show that $(x, y, z) = (0, 0, 0)$ is a unique non-negative integer solution of the Diophantine equation $3^x - p^y = z^2$ where p is prime and x, y, z are non-negative integers satisfying some conditions. For the case $p = 2$, we give all non-negative integer solutions of this equation.

Keywords: Diophantine equation, Mihailescu's Theorem, Non-negative integer solution

Introduction

Many mathematical researchers investigated the non-negative integer solutions $(x, y, z)$of Diophantine equations in the form $a^x - b^y = z^2$ where a and b are positive integers. In 2018, Rabago (2018) showed that the Diophantine equation $4^x - p^y = z^2$, where p is prime, has the set of all non-negative integer solutions $\{(x, y, z)\}$ given by $\{(x, y, z)\} = \{(0, 0, 0)\} \cup \{(q - 1, 1, 2^{q-1} - 1)\}$, for prime $p = 2^q - 1$ (with q also a prime). For $p \equiv 3 \pmod 4$ not of the form $2^q - 1$, the Diophantine equation $4^x - p^y = z^2$ has the only non-negative integer solution $(x, y, z) = (0, 0, 0)$. After that, in 2019, Thongnak, Chuayjan and Kaewong (2019) proved that $(x, y, z) \in \{(0, 0, 0), (1, 0, 1), (2, 1, 1)\}$ are only three non-negative integer solutions of the Diophantine equation $2^x - 3^y = z^2$.

In 2020, Burshtein (2020) proved that the Diophantine equation $13^x - 5^y = z^2$ has a unique positive integer solution $(x, y, z) = (2, 2, 12)$ and the Diophantine equation $19^x - 5^y = z^2$ has no positive integer solution. Recently, Thongnak, Chuayjan and Kaewong (2021, 2022) proved that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the Diophantine equations $7^x - 5^y = z^2$ and $7^x - 2^y = z^2$. In 2022, Tadee (2022) found all positive integer solutions of the Diophantine equation $p^{2x} - q^{2y} = z^2$ where p and q

are primes. Furthermore, Tadee and Laomalaw (2022) studied non-negative integer solutions of the Diophantine equation $2^x - p^y = z^2$ where p is prime.

In this paper, we study non-negative integer solutions of the Diophantine equation $3^x - p^y = z^2$ where p is prime and x, y, z are non-negative integers with some conditions.

Preliminaries

In this section, we give some helpful Theorems and Lemmas for this study.

Theorem 1. (Mihailescu's Theorem) (Mihailescu, 2004) The Diophantine equation $a^x - b^y = 1$ has a unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$ where a, b, x, y are integers and $\min\{a, b, x, y\} > 1$.

Theorem 2. (Tadee & Laomalaw, 2022) Let n be a positive integer with $n \neq 1$. Then the Diophantine equation $n^x - n^y = z^2$ has all non-negative integer solutions in the following form

$$(x, y, z) \in \left\{(r, r, 0), \left(1, 0, \sqrt{n-1}\right), \left(r+1, r, \sqrt{(n-1)n^r}\right)\right\} \cap \mathbb{Z}^3,$$

where r is a non-negative integer.

By Theorem 2, we have the following corollary for case $n = 3$.

Corollary 3. The Diophantine equation $3^x - 3^y = z^2$ has all non-negative integer solutions in the following form $(x, y, z) \in \{(r, r, 0) : r \in \mathbb{N} \cup \{0\}\}$.

Lemma 4. Let p be prime. Then the Diophantine equation $1 - p^y = z^2$ has a unique non-negative integer solution $(y, z) = (0,0)$.

Proof. Let y and z be non-negative integers such that $1 - p^y = z^2$. Since $z^2 \geq 0$, we have $1 - p^y \geq 0$. This implies that $y = 0$ and $z = 0$. ■

Lemma 5. The Diophantine equation $3^x - 1 = z^2$ has a unique non-negative integer solution $(x, z) = (0,0)$.

Proof. Let x and z be non-negative integers such that $3^x - 1 = z^2$. If $x = 0$, then $z^2 = 0$. It follows that $(x, z) = (0,0)$. If $x = 1$, then we have $z^2 = 2$ which is impossible. Assume that $x > 1$. We consider the following cases.

Case 1. $z = 0$. Then $3^x = 1$. We get $x = 0$. This is impossible since $x > 1$.

Case 2. $z = 1$. Then $3^x = 2$. This is impossible since x is an integer.

Case 3. $z > 1$. Then $\min\{3, z, x, 2\} > 1$. This is impossible since $3^x - z^2 = 1$ and Theorem 1. ■

Lemma 6. Let p be prime and x be an even integer. If the Diophantine equation $3^x - p^y = z^2$ has a non-negative integer solution, then there exists a non-negative integer u such that $2 \cdot 3^{\frac{x}{2}} = p^u(p^{y-2u} + 1)$.

Proof. Let x, y and z be non-negative integers such that $3^x - p^y = z^2$. Since x is an even integer, there exists a non-negative integer k such that $x = 2k$. Then $(3^k - z)(3^k + z) = p^y$. Since p is prime, there exists a non-negative integer u such that $3^k - z = p^u$ and $3^k + z = p^{y-u}$. Thus, $y \geq 2u$ and $2 \cdot 3^k = p^u(p^{y-2u} + 1)$. ■

Corollary 7. Let $p \notin \{2,3\}$ be prime and x be an even integer. Then all non-negative integer solutions of the Diophantine equation $3^x - p^y = z^2$ are

$$(x,y,z) \in \left\{\left(2k,\ \log_p(2\cdot 3^k - 1), 3^k - 1\right): k,\ \log_p(2\cdot 3^k - 1) \in \mathbb{N} \cup \{0\}\right\}.$$

Proof. Let x, y and z be non-negative integers such that $3^x - p^y = z^2$. Since x is an even integer, there exists a non-negative integer k such that $x = 2k$. By Lemma 6, there exists a non-negative integer u such that $2 \cdot 3^k = p^u(p^{y-2u} + 1)$. Since $p \notin \{2,3\}$ is prime, we obtain that $u = 0$. Then $2 \cdot 3^k = p^y + 1$. Thus, $y = \log_p(2 \cdot 3^k - 1)$ and $z^2 = 3^x - p^y = 3^{2k} - 2 \cdot 3^k + 1 = (3^k - 1)^2$. This implies that $z = 3^k - 1$. Then $(x,y,z) = \left(2k,\ \log_p(2\cdot 3^k - 1), 3^k - 1\right)$ where $\log_p(2\cdot 3^k - 1)$ is a non-negative integer. ■

Corollary 8. The Diophantine equation $81^x - 5^y = z^2$ has a unique non-negative integer solution $(x,y,z) = (0,0,0)$.

Proof. Let x, y and z be non-negative integers such that $3^{4x} - 5^y = z^2$. Assume that $y \geq 1$. Since $4x$ is an even integer and Lemma 6, there exists a non-negative integer u such that $2 \cdot 9^x = 5^u(5^{y-2u} + 1)$. Then $u = 0$ and $2 \cdot 9^x = 5^y + 1$. Since $5^y + 1 \equiv 1 (\mathrm{mod}\ 5)$, we get $2 \cdot 9^x \equiv 1 (\mathrm{mod}\ 5)$. This is impossible since $2 \cdot 9^x \equiv 2 \cdot (-1)^x \equiv -2$ or $2 (\mathrm{mod}\ 5)$. Thus, $y = 0$. By Lemma 5, we have $(x,y,z) = (0,0,0)$. ■

Main Results

We now present our main results.

Theorem 9. Let p be prime with $p \equiv 1 (\mathrm{mod}\ 3)$. Then the Diophantine equation $3^x - p^y = z^2$ has a unique non-negative integer solution $(x,y,z) = (0,0,0)$.

Proof. Let x, y and z be non-negative integers such that $3^x - p^y = z^2$. Assume that $x \geq 1$. Then $3^x \equiv 0 (\mathrm{mod}\ 3)$. Since $p \equiv 1 (\mathrm{mod}\ 3)$, we obtain that $3^x - p^y \equiv -1 (\mathrm{mod}\ 3)$. Then $z^2 \equiv -1 (\mathrm{mod}\ 3)$, which contradicts the fact that $z^2 \equiv 0$ or $1 (\mathrm{mod}\ 3)$. Thus, $x = 0$. By Lemma 4, we have $(x,y,z) = (0,0,0)$. ■

Theorem 10. Let p be prime with $p \equiv 2 (\mathrm{mod}\ 3)$ and y be an even integer. Then the Diophantine equation $3^x - p^y = z^2$ has a unique non-negative integer solution$(x,y,z) = (0,0,0)$.

Proof. Let x, y and z be non-negative integers such that $3^x - p^y = z^2$. If $x \geq 1$, then $3^x \equiv 0 (\mathrm{mod}\ 3)$. Since $p \equiv 2 (\mathrm{mod}\ 3)$ and y is even, we get $3^x - p^y \equiv 0 - (-1)^y \equiv -1 (\mathrm{mod}\ 3)$. This implies that $z^2 \equiv -1 (\mathrm{mod}\ 3)$, which contradicts the fact that $z^2 \equiv 0$ or $1 (\mathrm{mod}\ 3)$. Thus $x = 0$. By Lemma 4, we have $(x,y,z) = (0,0,0)$. ■

Corollary 11. Let p be prime with $p \equiv 29 (\mathrm{mod}\ 60)$. Then the Diophantine equation $3^x - p^y = z^2$ has a unique non-negative integer solution $(x,y,z) = (0,0,0)$.

Proof. Let x, y and z be non-negative integers such that $3^x - p^y = z^2$. Since $p \equiv 29 (\mathrm{mod}\ 60)$, we have $p \equiv 2 (\mathrm{mod}\ 3)$, $p \equiv 1 (\mathrm{mod}\ 4)$ and $p \equiv -1 (\mathrm{mod}\ 5)$. Assume that y is odd. Since $p \equiv 1 (\mathrm{mod}\ 4)$, we get $z^2 = 3^x -$

$p^y \equiv (-1)^x - 1 (\mathrm{mod}\ 4)$. Since p is odd, we have $3^x - p^y$ is even. Thus, z is also even. Then $z^2 \equiv 0 (\mathrm{mod}\ 4)$. This implies that $(-1)^x - 1 \equiv 0 (\mathrm{mod}\ 4)$. Then x is even. By Lemma 6, there exists a non-negative integer u such that $2 \cdot 3^{\frac{x}{2}} = p^u(p^{y-2u} + 1)$. Since p is prime with $p \equiv 29 (\mathrm{mod}\ 60)$, we have $u = 0$ and $2 \cdot 3^{\frac{x}{2}} = p^y + 1$. Since $p \equiv -1 (\mathrm{mod}\ 5)$ and y is odd, we have $p^y + 1 \equiv 0 (\mathrm{mod}\ 5)$. Then $2 \cdot 3^{\frac{x}{2}} \equiv 0 (\mathrm{mod}\ 5)$. This is a contradiction. Thus, y is even. Since $p \equiv 2 (\mathrm{mod}\ 3)$ and Theorem 10, we have $(x, y, z) = (0, 0, 0)$. ■

Theorem 12. Let p be prime with $p \equiv 5 (\mathrm{mod}\ 12)$. If there exists a prime $q \notin \{2, 3\}$ such that $p \equiv -1 (\mathrm{mod}\ q)$, then the Diophantine equation $3^x - p^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (0, 0, 0)$.

Proof. Let x, y and z be non-negative integers such that $3^x - p^y = z^2$. Since $p \equiv 5 (\mathrm{mod}\ 12)$, we have $p \equiv 2 (\mathrm{mod}\ 3)$ and $p \equiv 1 (\mathrm{mod}\ 4)$. Assume that y is odd. Since $p \equiv 1 (\mathrm{mod}\ 4)$, we have $z^2 = 3^x - p^y \equiv (-1)^x - 1 (\mathrm{mod}\ 4)$. Since p is odd, we obtain that $3^x - p^y$ is even. Thus, z is even and $z^2 \equiv 0 (\mathrm{mod}\ 4)$. It follows that $(-1)^x - 1 \equiv 0 (\mathrm{mod}\ 4)$. Then x is even. By Lemma 6, there exists a non-negative integer u such that $2 \cdot 3^{\frac{x}{2}} = p^u(p^{y-2u} + 1)$. Since p is prime with $p \equiv 5 (\mathrm{mod}\ 12)$, we get $u = 0$ and $2 \cdot 3^{\frac{x}{2}} = p^y + 1$. Since $p \equiv -1 (\mathrm{mod}\ q)$ and y is odd, we have $p^y + 1 \equiv 0 (\mathrm{mod}\ q)$. Thus, $2 \cdot 3^{\frac{x}{2}} \equiv 0 (\mathrm{mod}\ q)$. This is impossible since $q \notin \{2, 3\}$ is prime. Thus, y is even. Since $p \equiv 2 (\mathrm{mod}\ 3)$ and Theorem 10, we get $(x, y, z) = (0, 0, 0)$. ■

Corollary 13. The Diophantine equation $3^x - 41^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (0, 0, 0)$.

Proof. Since 41 is prime, $41 \equiv 5 (\mathrm{mod}\ 12)$ and $41 \equiv -1 (\mathrm{mod}\ 7)$, we obtain that the Diophantine equation $3^x - 41^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (0, 0, 0)$, by Theorem 12. ■

Theorem 14. Let p be prime with $p \equiv 11 (\mathrm{mod}\ 12)$ and x be an even integer. Then the Diophantine equation $3^x - p^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (0, 0, 0)$.

Proof. Let x, y and z be non-negative integers such that $3^x - p^y = z^2$. Since x is even and Lemma 6, there exists a non-negative integer u such that $2 \cdot 3^{\frac{x}{2}} = p^u(p^{y-2u} + 1)$. Since p is prime with $p \equiv 11 (\mathrm{mod}\ 12)$, we get $u = 0$ and $2 \cdot 3^{\frac{x}{2}} = p^y + 1$. Since $p \equiv -1 (\mathrm{mod}\ 4)$, we have $p^y + 1 \equiv (-1)^y + 1 (\mathrm{mod}\ 4)$. Since $3 \equiv -1 (\mathrm{mod}\ 4)$, we have $2 \cdot 3^{\frac{x}{2}} \equiv 2(-1)^x \equiv -2$ or $2 (\mathrm{mod}\ 4)$. Thus, $(-1)^y + 1 \equiv -2$ or $2 (\mathrm{mod}\ 4)$. Then y is even. Since $p \equiv 2 (\mathrm{mod}\ 3)$ and Theorem 10, we get $(x, y, z) = (0, 0, 0)$. ■

Theorem 15. The Diophantine equation $3^x - 2^y = z^2$ has all non-negative integer solutions in the following form $(x, y, z) \in \{(0, 0, 0), (2, 3, 1), (4, 5, 7)\} \cup \{(r, 1, \sqrt{3^r - 2}) : r, \sqrt{3^r - 2} \in \mathbb{N}\}$.

Proof. Let x, y and z be non-negative integers such that $3^x - 2^y = z^2$. If $y = 0$, then by Lemma 5, we have $(x, y, z) = (0, 0, 0)$. If $y = 1$, then $z = \sqrt{3^x - 2}$. This implies that $(x, y, z) \in \{(r, 1, \sqrt{3^r - 2}) : r, \sqrt{3^r - 2} \in \mathbb{N}\}$. If $y > 1$, then we get $x > 1$ and $3^x - 2^y \equiv (-1)^x (\mathrm{mod}\ 4)$. Thus, $z^2 \equiv (-1)^x (\mathrm{mod}\ 4)$. Since $3^x - 2^y$ is odd, we have z^2 is also odd. Then $z^2 \equiv 1 (\mathrm{mod}\ 4)$. Thus, $(-1)^x \equiv 1 (\mathrm{mod}\ 4)$. Then x is even. By Lemma 6, there exists a non-negative integer u such that $2 \cdot 3^{\frac{x}{2}} = 2^u(2^{y-2u} + 1)$. Then $u = 1$ and $3^{\frac{x}{2}} = 2^{y-2} + 1$.

Case 1. $x = 2$. Then $2^{y-2} = 2$. Thus, $y = 3$ and $z = 1$. That is $(x, y, z) = (2, 3, 1)$.

Case 2. $x \geq 4$. If $y = 2$, then $3^{\frac{x}{2}} = 2$. This is impossible. If $y = 3$, then $3^{\frac{x}{2}} = 3$ and $x = 2$. This is also impossible. Thus, $y \geq 4$. Then $\min\left\{3, 2, \frac{x}{2}, y-2\right\} > 1$. Since $3^{\frac{x}{2}} - 2^{y-2} = 1$ and Theorem 1, we get $x = 4$ and $y = 5$. Then $z^2 = 3^4 - 2^5 = 49$. This implies that $z = 7$. That is $(x, y, z) = (4, 5, 7)$. ■

Remark: By Theorem 15, $\{(1,1,1), (3,1,5), (14,1,2{,}187), (16,1,6{,}561)\}$ are non-negative integer solutions of the Diophantine equation $3^x - 2^y = z^2$.

Conclusions

In this article, we proved that the Diophantine equation $3^x - p^y = z^2$, where p is prime and x, y, z are non-negative integers satisfying some conditions, has a unique non-negative integer solution $(x, y, z) = (0, 0, 0)$. For $p = 2$, this Diophantine equation has all non-negative integer solutions in the following form

$$(x, y, z) \in \{(0,0,0), (2,3,1), (4,5,7)\} \cup \left\{\left(r, 1, \sqrt{3^r - 2}\right) : r, \sqrt{3^r - 2} \in \mathbb{N}\right\}.$$

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